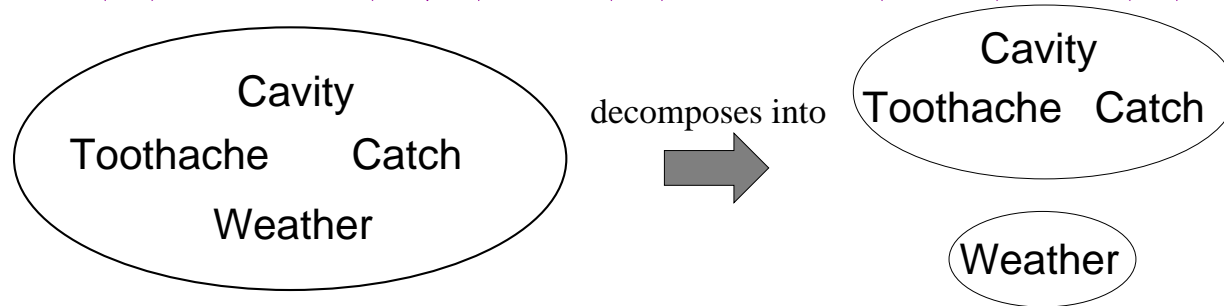


Independence

A and B are independent iff

$$\mathbf{P}(A|B) = \mathbf{P}(A) \quad \text{or} \quad \mathbf{P}(B|A) = \mathbf{P}(B) \quad \text{or} \quad \mathbf{P}(A, B) = \mathbf{P}(A)\mathbf{P}(B)$$



$$\begin{aligned} &\mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity}, \textit{Weather}) \\ &= \mathbf{P}(\textit{Toothache}, \textit{Catch}, \textit{Cavity})\mathbf{P}(\textit{Weather}) \end{aligned}$$

32 entries reduced to 12; for n independent biased coins, $2^n \rightarrow n$

Absolute independence powerful but rare

Dentistry is a large field with hundreds of variables,
none of which are independent. What to do?

Conditional independence

$\mathbf{P}(\textit{Toothache}, \textit{Cavity}, \textit{Catch})$ has $2^3 - 1 = 7$ independent entries

If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:

$$(1) \ P(\textit{catch}|\textit{toothache}, \textit{cavity}) = P(\textit{catch}|\textit{cavity})$$

The same independence holds if I haven't got a cavity:

$$(2) \ P(\textit{catch}|\textit{toothache}, \neg\textit{cavity}) = P(\textit{catch}|\neg\textit{cavity})$$

Catch is **conditionally independent** of *Toothache* given *Cavity*:

$$\mathbf{P}(\textit{Catch}|\textit{Toothache}, \textit{Cavity}) = \mathbf{P}(\textit{Catch}|\textit{Cavity})$$

Equivalent statements:

$$\mathbf{P}(\textit{Toothache}|\textit{Catch}, \textit{Cavity}) = \mathbf{P}(\textit{Toothache}|\textit{Cavity})$$

$$\mathbf{P}(\textit{Toothache}, \textit{Catch}|\textit{Cavity}) = \mathbf{P}(\textit{Toothache}|\textit{Cavity})\mathbf{P}(\textit{Catch}|\textit{Cavity})$$

Bayesian networks

A simple, graphical notation for conditional independence assertions and hence for compact specification of full joint distributions

Syntax:

- a set of nodes, one per variable

- a directed, acyclic graph (link \approx “directly influences”)

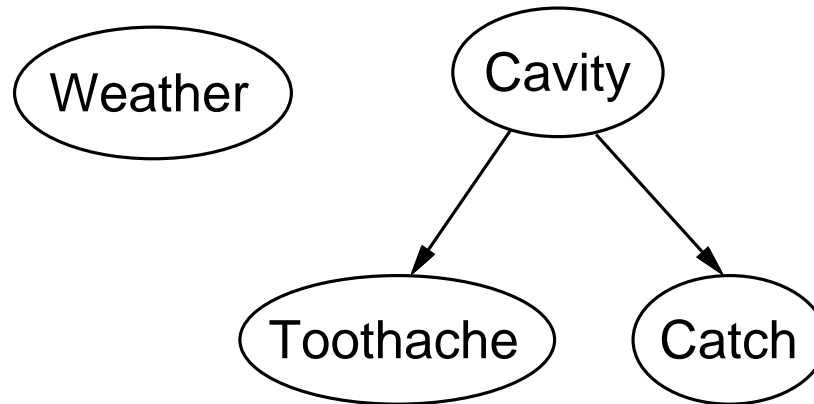
- a conditional distribution for each node given its parents:

$$\mathbf{P}(X_i | Parents(X_i))$$

In the simplest case, conditional distribution represented as a **conditional probability table** (CPT) giving the distribution over X_i for each combination of parent values

Example

Topology of network encodes conditional independence assertions:



Weather is independent of the other variables

Toothache and *Catch* are conditionally independent given *Cavity*

Example

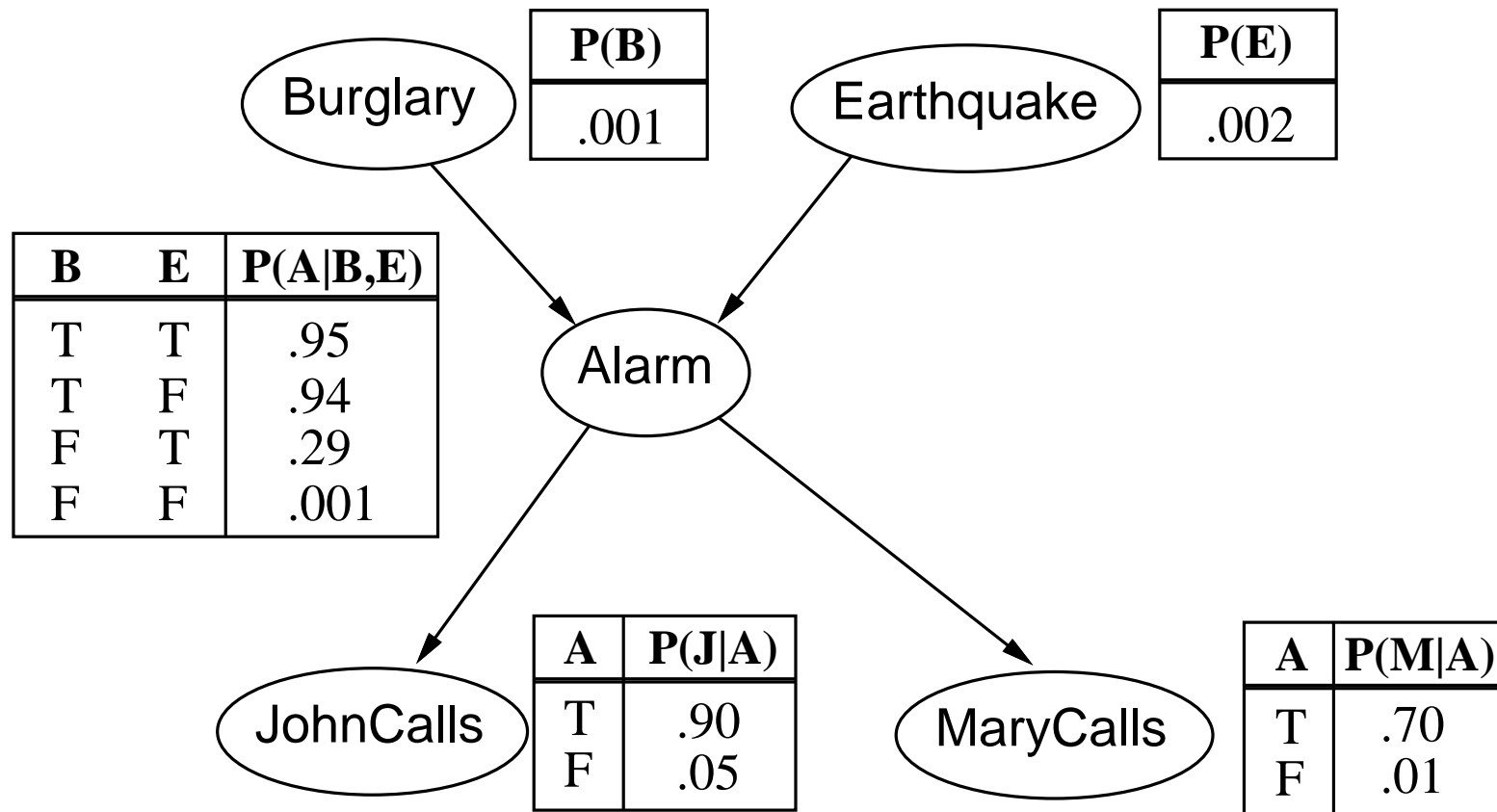
I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

Variables: *Burglar*, *Earthquake*, *Alarm*, *JohnCalls*, *MaryCalls*

Network topology reflects “causal” knowledge:

- A burglar can set the alarm off
- An earthquake can set the alarm off
- The alarm can cause Mary to call
- The alarm can cause John to call

Example contd.



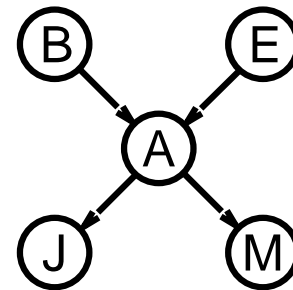
Global semantics

Global semantics defines the full joint distribution as the product of the local conditional distributions:

$$P(x_1, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

e.g., $P(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

=



Global semantics

“Global” semantics defines the full joint distribution as the product of the local conditional distributions:

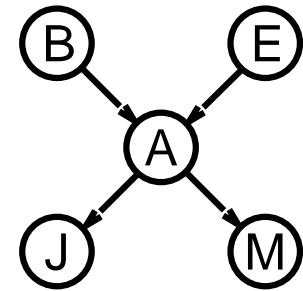
$$P(x_1, \dots, x_n) = \prod_{i=1}^n P(x_i | \text{parents}(X_i))$$

e.g., $P(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

$$= P(j|a)P(m|a)P(a|\neg b, \neg e)P(\neg b)P(\neg e)$$

$$= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998$$

$$\approx 0.00063$$



Inference by stochastic simulation

Basic idea:

- 1) Draw N samples from a sampling distribution S
- 2) Compute an approximate posterior probability \hat{P}
- 3) Show this converges to the true probability P

0.5

Coin

Outline:

- Sampling from an empty network
- Rejection sampling: reject samples disagreeing with evidence
- Likelihood weighting: use evidence to weight samples
- Markov chain Monte Carlo (MCMC): sample from a stochastic process whose stationary distribution is the true posterior

Approximate inference using MCMC

“State” of network = current assignment to all variables.

Generate next state by sampling one variable given Markov blanket

Sample each variable in turn, keeping evidence fixed

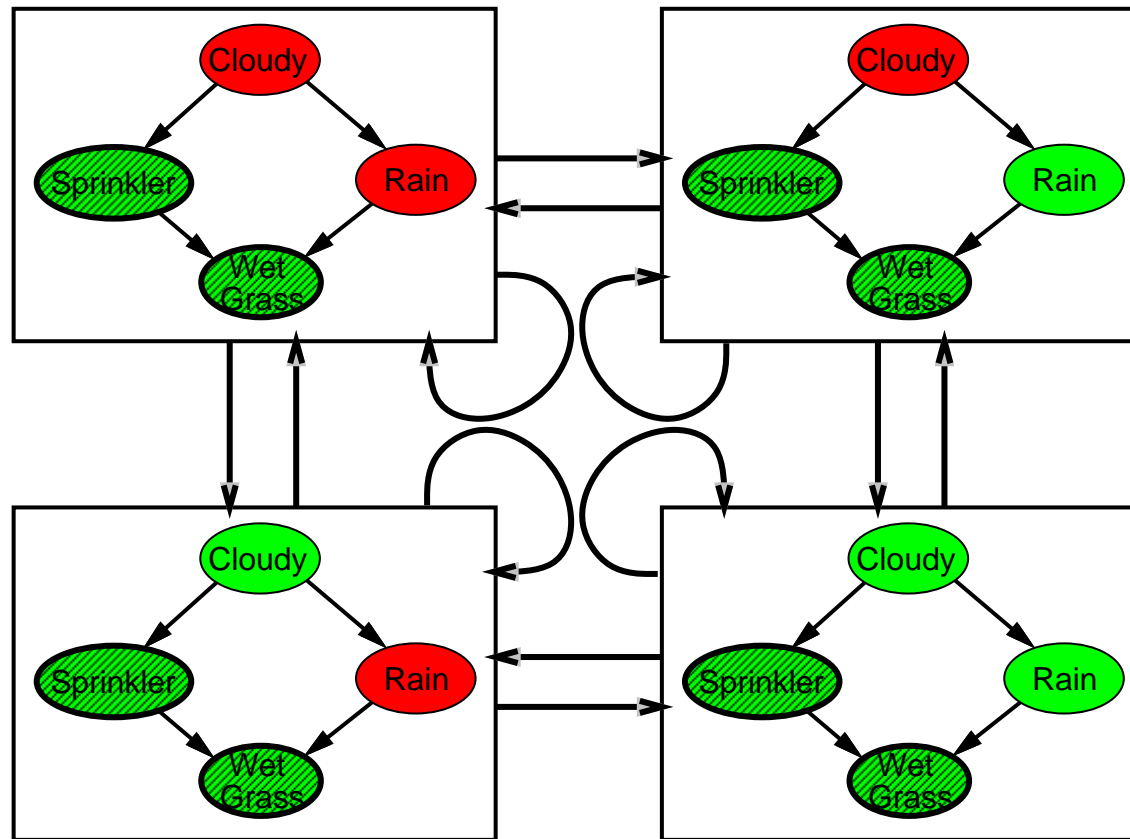
```
function MCMC-Ask( $X, \mathbf{e}, bn, N$ ) returns an estimate of  $P(X|\mathbf{e})$ 
  local variables:  $\mathbf{N}[X]$ , a vector of counts over  $X$ , initially zero
                   $\mathbf{Z}$ , the nonevidence variables in  $bn$ 
                   $\mathbf{x}$ , the current state of the network, initially copied from  $\mathbf{e}$ 

  initialize  $\mathbf{x}$  with random values for the variables in  $\mathbf{Y}$ 
  for  $j = 1$  to  $N$  do
    for each  $Z_i$  in  $\mathbf{Z}$  do
      sample the value of  $Z_i$  in  $\mathbf{x}$  from  $P(Z_i|mb(Z_i))$ 
        given the values of  $MB(Z_i)$  in  $\mathbf{x}$ 
       $\mathbf{N}[x] \leftarrow \mathbf{N}[x] + 1$  where  $x$  is the value of  $X$  in  $\mathbf{x}$ 
  return NORMALIZE( $\mathbf{N}[X]$ )
```

Can also choose a variable to sample at random each time

The Markov chain

With *Sprinkler* = *true*, *WetGrass* = *true*, there are four states:



Wander about for a while, average what you see

MCMC example contd.

Estimate $\mathbf{P}(Rain|Sprinkler = true, WetGrass = true)$

Sample *Cloudy* or *Rain* given its Markov blanket, repeat.
Count number of times *Rain* is true and false in the samples.

E.g., visit 100 states

31 have *Rain = true*, 69 have *Rain = false*

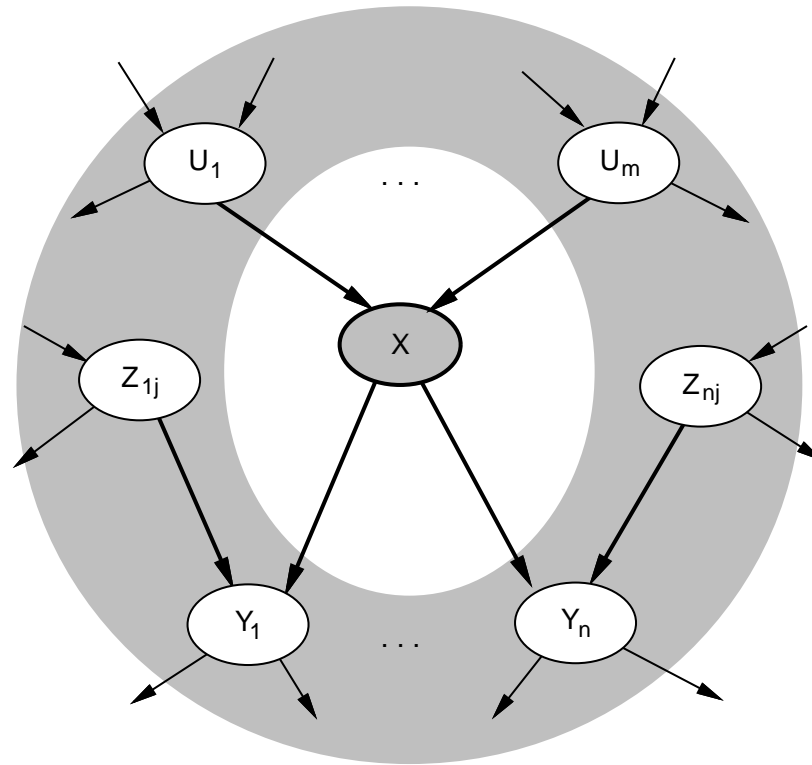
$$\begin{aligned}\hat{\mathbf{P}}(Rain|Sprinkler = true, WetGrass = true) \\ = \text{NORMALIZE}(\langle 31, 69 \rangle) = \langle 0.31, 0.69 \rangle\end{aligned}$$

Theorem: chain approaches **stationary distribution**:

long-run fraction of time spent in each state is exactly
proportional to its posterior probability

Markov blanket

Each node is conditionally independent of all others given its
Markov blanket: parents + children + children's parents



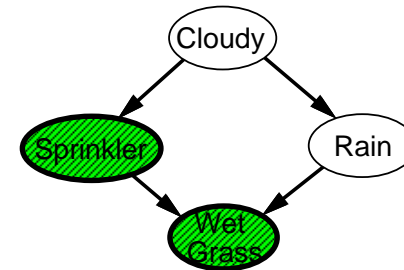
Markov blanket sampling

Markov blanket of *Cloudy* is

Sprinkler and *Rain*

Markov blanket of *Rain* is

Cloudy, *Sprinkler*, and *WetGrass*



Probability given the Markov blanket is calculated as follows:

$$P(x'_i | mb(X_i)) = P(x'_i | parents(X_i)) \prod_{Z_j \in Children(X_i)} P(z_j | parents(Z_j))$$

Easily implemented in message-passing parallel systems, brains

Main computational problems:

- 1) Difficult to tell if convergence has been achieved
- 2) Can be wasteful if Markov blanket is large:

$P(X_i | mb(X_i))$ won't change much (law of large numbers)

Constructing Bayesian networks

Need a method such that a series of locally testable assertions of conditional independence guarantees the required global semantics

1. Choose an ordering of variables X_1, \dots, X_n
2. For $i = 1$ to n
 add X_i to the network
 select parents from X_1, \dots, X_{i-1} such that
 $\mathbf{P}(X_i | \text{Parents}(X_i)) = \mathbf{P}(X_i | X_1, \dots, X_{i-1})$

This choice of parents guarantees the global semantics:

$$\begin{aligned}\mathbf{P}(X_1, \dots, X_n) &= \prod_{i=1}^n \mathbf{P}(X_i | X_1, \dots, X_{i-1}) \quad (\text{chain rule}) \\ &= \prod_{i=1}^n \mathbf{P}(X_i | \text{Parents}(X_i)) \quad (\text{by construction})\end{aligned}$$

Example

Suppose we choose the ordering M, J, A, B, E

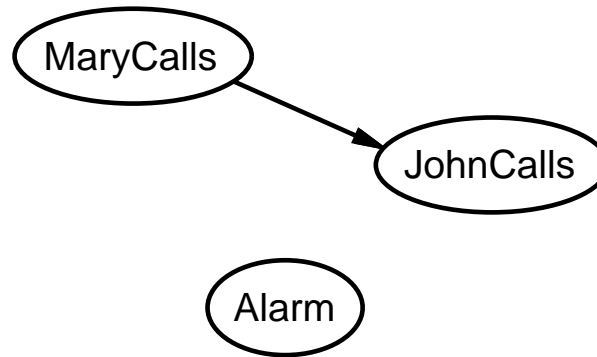
MaryCalls

JohnCalls

$$P(J|M) = P(J)?$$

Example

Suppose we choose the ordering M, J, A, B, E

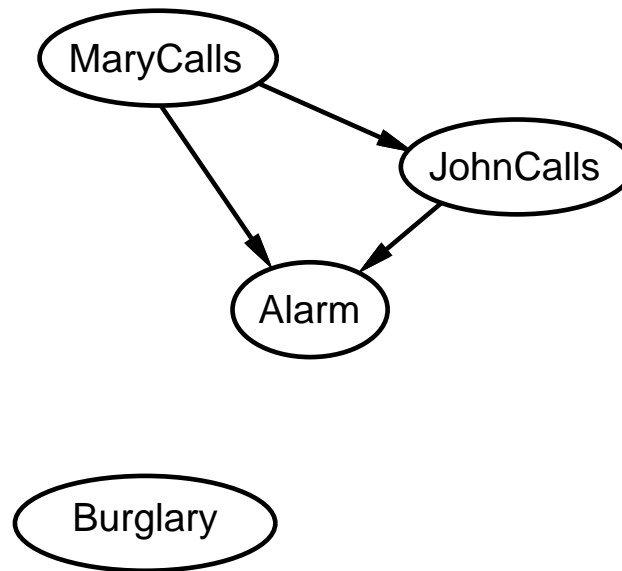


$P(J|M) = P(J)$? No

$P(A|J, M) = P(A|J)$? $P(A|J, M) = P(A)$?

Example

Suppose we choose the ordering M, J, A, B, E



$P(J|M) = P(J)$? No

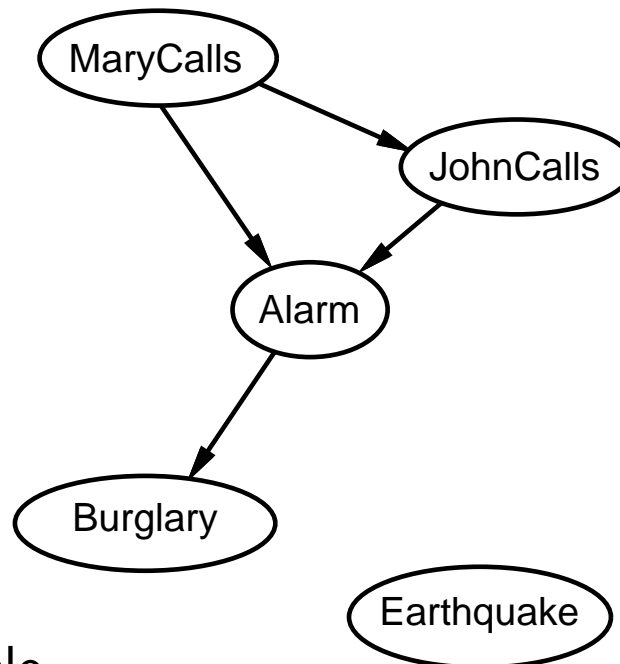
$P(A|J, M) = P(A|J)$? $P(A|J, M) = P(A)$? No

$P(B|A, J, M) = P(B|A)$?

$P(B|A, J, M) = P(B)$?

Example

Suppose we choose the ordering M, J, A, B, E



$P(J|M) = P(J)$? No

$P(A|J, M) = P(A|J)$? $P(A|J, M) = P(A)$? No

$P(B|A, J, M) = P(B|A)$? Yes

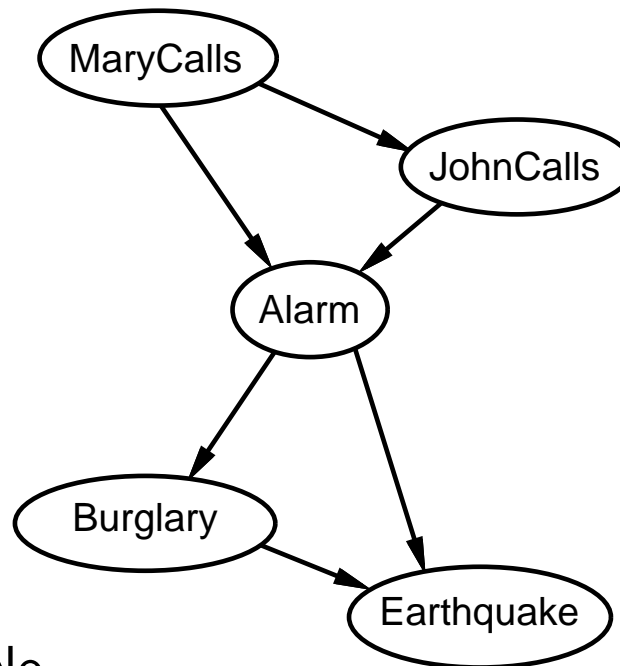
$P(B|A, J, M) = P(B)$? No

$P(E|B, A, J, M) = P(E|A)$?

$P(E|B, A, J, M) = P(E|A, B)$?

Example

Suppose we choose the ordering M, J, A, B, E



$P(J|M) = P(J)$? No

$P(A|J, M) = P(A|J)$? $P(A|J, M) = P(A)$? No

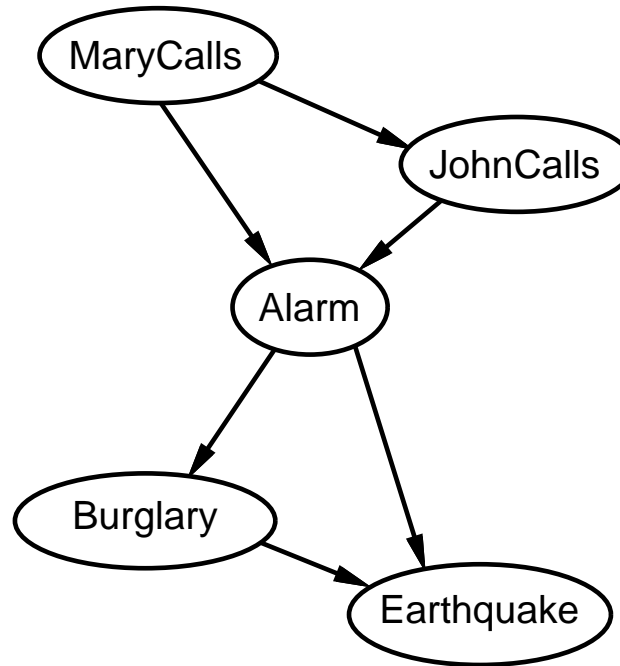
$P(B|A, J, M) = P(B|A)$? Yes

$P(B|A, J, M) = P(B)$? No

$P(E|B, A, J, M) = P(E|A)$? No

$P(E|B, A, J, M) = P(E|A, B)$? Yes

Example contd.



Deciding conditional independence is hard in noncausal directions

(Causal models and conditional independence seem hardwired for humans!)

Assessing conditional probabilities is hard in noncausal directions

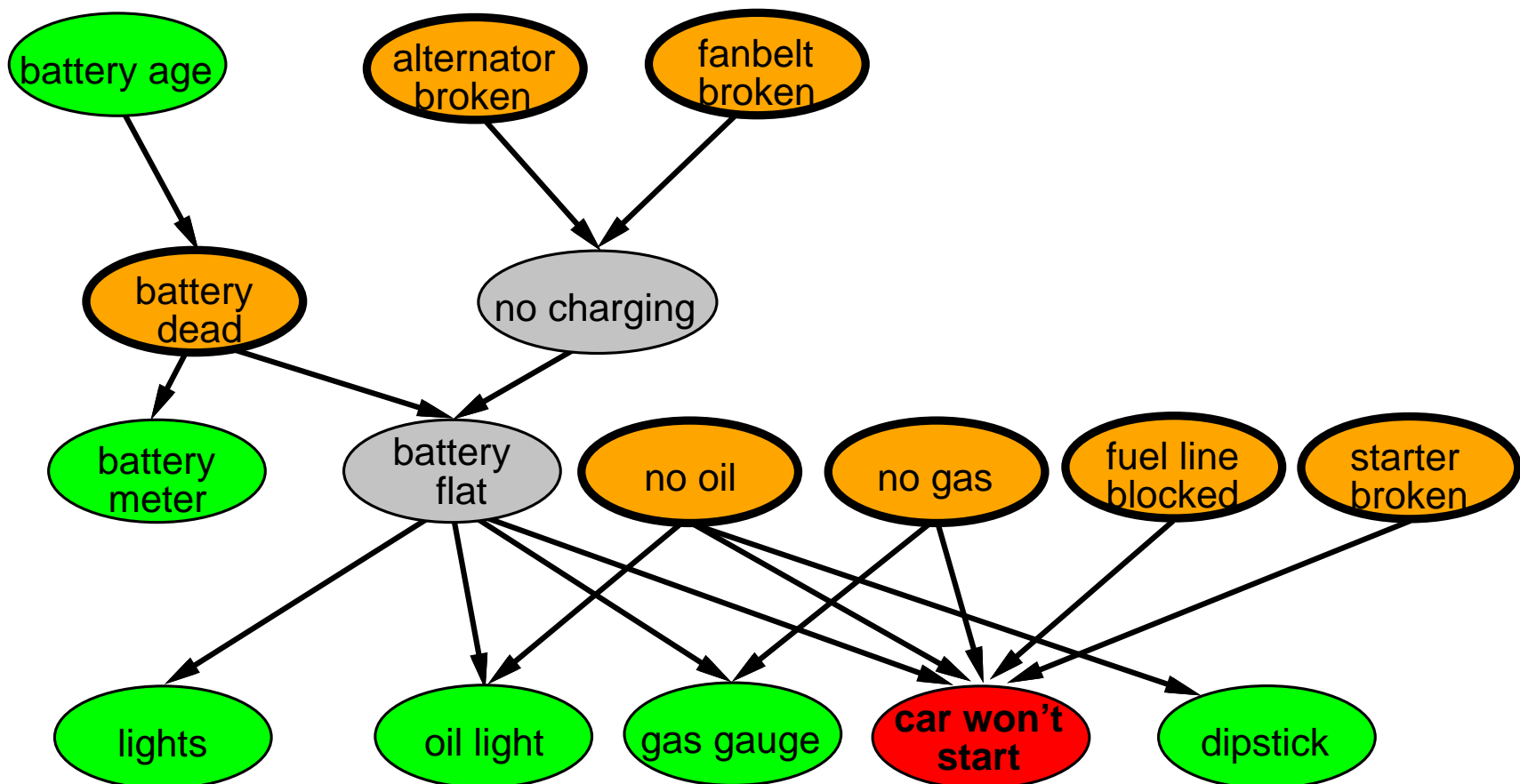
Network is less compact: $1 + 2 + 4 + 2 + 4 = 13$ numbers needed

Example: Car diagnosis

Initial evidence: car won't start

Testable variables (green), "broken, so fix it" variables (orange)

Hidden variables (gray) ensure sparse structure, reduce parameters



Compact conditional distributions

CPT grows exponentially with number of parents

CPT becomes infinite with continuous-valued parent or child

Solution: **canonical** distributions that are defined compactly

Deterministic nodes are the simplest case:

$$X = f(\text{Parents}(X)) \text{ for some function } f$$

E.g., Boolean functions

$$\text{NorthAmerican} \Leftrightarrow \text{Canadian} \vee \text{US} \vee \text{Mexican}$$

E.g., numerical relationships among continuous variables

$$\frac{\partial \text{Level}}{\partial t} = \text{inflow} + \text{precipitation} - \text{outflow} - \text{evaporation}$$

Compact conditional distributions contd.

Noisy-OR distributions model multiple noninteracting causes

- 1) Parents $U_1 \dots U_k$ include all causes (can add **leak node**)
- 2) Independent failure probability q_i for each cause alone

$$\Rightarrow P(X|U_1 \dots U_j, \neg U_{j+1} \dots \neg U_k) = 1 - \prod_{i=1}^j q_i$$

<i>Cold</i>	<i>Flu</i>	<i>Malaria</i>	$P(\text{Fever})$	$P(\neg \text{Fever})$
F	F	F	0.0	1.0
F	F	T	0.9	0.1
F	T	F	0.8	0.2
F	T	T	0.98	$0.02 = 0.2 \times 0.1$
T	F	F	0.4	0.6
T	F	T	0.94	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.012 = 0.6 \times 0.2 \times 0.1$

Number of parameters **linear** in number of parents

Inference tasks

Simple queries: compute posterior marginal $\mathbf{P}(X_i|\mathbf{E} = \mathbf{e})$

e.g., $P(\text{NoGas} | \text{Gauge} = \text{empty}, \text{Lights} = \text{on}, \text{Starts} = \text{false})$

Conjunctive queries: $\mathbf{P}(X_i, X_j | \mathbf{E} = \mathbf{e}) = \mathbf{P}(X_i | \mathbf{E} = \mathbf{e}) \mathbf{P}(X_j | X_i, \mathbf{E} = \mathbf{e})$

Optimal decisions: decision networks include utility information;
probabilistic inference required for $P(\text{outcome} | \text{action}, \text{evidence})$

Value of information: which evidence to seek next?

Sensitivity analysis: which probability values are most critical?

Explanation: why do I need a new starter motor?